

# $C^{1,\alpha}$ $h$ -principle for von Kármán constraints

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## Abstract

Exploiting some connections between solutions  $v : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $w : \Omega \rightarrow \mathbb{R}^2$  of the system  $\nabla v \otimes \nabla v + 2 \operatorname{sym} \nabla w = A$  and the isometric immersion problem in two dimensions, we provide a simple construction of  $C^{1,\alpha}$  convex integration solutions for the former from the corresponding result for the latter.

**Keywords:**  $h$ -principle, isometric embeddings, Monge-Ampère, von Kármán, nonlinear elasticity

**Mathematics Subject Classification:** 35J96, 74G20

## 1 Introduction and main result

The classical  $h$ -principle of Nash and Kuiper shows that there exist surprisingly many  $C^1$  solutions  $u : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$  to the isometric immersion system

$$(\nabla u)^T(\nabla u) = g. \quad (1)$$

In contrast, classical rigidity results show that, among more regular immersions, being a solution of system (1) is as restrictive a condition as one might expect. A natural question is whether such results extend to classes of  $C^{1,\alpha}$  Hölder spaces: for the  $h$ -principle one seeks the largest possible Hölder exponent  $\alpha \in (0, 1)$  and for the rigidity the smallest possible one. We refer to [1] and the references therein.

Such a dichotomy between an  $h$ -principle on one hand and rigidity on the

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other hand also applies to other PDE systems. A system for which this is to be expected is the system

$$\nabla v \otimes \nabla v + 2 \operatorname{sym} \nabla w = A \quad (2)$$

for  $w : \Omega \rightarrow \mathbb{R}^2$  and  $v : \Omega \rightarrow \mathbb{R}$ . This system arises naturally as a constraint in von Kármán theories (cf. [3]) in certain energy regimes. In that context,  $w$  describes the in-plane displacement and  $v$  the out-of-plane displacement. It is clearly related to the Monge-Ampère equation  $\det \nabla^2 v = \operatorname{curl} \operatorname{curl} A$ . We refer e.g. to [3] for some details on this.

System (2) is closely related to (1), and it was shown in [4] that the convex integration construction in [1] can indeed be adapted to obtain the same statement for system (2). In this paper we show how the close connection between (2) and (1) can be used to derive  $C^{1,\alpha}$   $h$ -principles for (2) directly from similar results for (1), without having to repeat the construction.

From now on  $\Omega \subset \mathbb{R}^2$  denotes a bounded and simply connected domain with a smooth boundary. Our main result is the following:

**Theorem 1.1.** *Let  $p \geq 2$ , let  $\beta \in (0, 1)$  and let*

$$0 < \alpha < \min \left\{ \frac{1}{7}, \frac{\beta}{2} \right\}.$$

*Then there exists  $C > 0$  such that the following is true:*

*For any  $v \in C^2(\overline{\Omega})$ ,  $w \in C^2(\overline{\Omega}, \mathbb{R}^2)$ ,  $A \in C^{0,\beta}(\overline{\Omega}, \mathbb{R}_{\operatorname{sym}}^{2 \times 2})$  there exist  $\overline{v} \in C^{1,\alpha}(\overline{\Omega})$  and  $\overline{w} \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^2)$  with*

$$\|\nabla \overline{v} - \nabla v\|_{C^0(\Omega)} \leq C \|\nabla v \otimes \nabla v + 2 \operatorname{sym} \nabla w - A\|_{C^0(\Omega)}^{1/2} \quad (3)$$

*and*

$$\begin{aligned} \|\nabla \overline{w} - \nabla w\|_{L^p(\Omega)} &\leq C \|\nabla v\|_{C^0(\Omega)} \|\nabla v \otimes \nabla v + 2 \operatorname{sym} \nabla w - A\|_{C^0(\Omega)}^{1/2} \\ &\quad + \|\nabla v \otimes \nabla v + 2 \operatorname{sym} \nabla w - A\|_{C^0(\Omega)}, \end{aligned} \quad (4)$$

*and such that*

$$2 \operatorname{sym} \nabla \overline{w} + \nabla \overline{v} \otimes \nabla \overline{v} = A. \quad (5)$$

**Remarks.**

1. Theorem 1.1 is a variant of [1, Theorem 1]. It allows to improve a  $C^1$   $h$ -principle to a  $C^{1,\alpha}$   $h$ -principle, cf. Corollary 3.2.

2. A variant of Theorem 1.1 was stated in [4]. Theorem 1.1 is more general in that does not require  $2 \operatorname{sym} \nabla w + \nabla v \otimes \nabla v$  to be close to  $A$ . On the other hand it only yields  $L^p$  rather than uniform bounds on  $\nabla w$ . (For the actual convex integration result, however, this is immaterial. See Corollary 3.2 below.) The main difference to [4] is that our short proof derives Theorem 1.1 directly from the corresponding result for isometric immersions [1], therefore avoiding the need of adapting each step of the construction in [1].

**Notation.** For  $n \in \mathbb{N}$ , we denote by  $\mathbb{R}_{\operatorname{sym}}^{n \times n}$  the set of symmetric  $n \times n$  matrices. By  $e$  we denote the standard Riemannian metric on  $\mathbb{R}^n$ . Given an immersion  $u$  into  $\mathbb{R}^n$ , we denote by  $u^*e$  the pullback-metric, so that in coordinates

$$(u^*e)_{ij} = \partial_i u \cdot \partial_j u.$$

For  $k = 0, 1, \dots$  we denote the usual  $C^k$  norm by  $\|u\|_k$ . For  $\beta \in (0, 1)$  the Hölder seminorm  $[u]_\beta$  is defined to be the infimum over all  $C$  such that

$$|u(x) - u(y)| \leq C|x - y|^\beta \text{ for all } x, y \in \Omega.$$

The unit matrix is denoted by  $I$ .

## 2 $h$ -principle for isometric immersions

An inspection of the proof in [1] shows that in that paper the following more detailed version of [1, Theorem 1] is proven:

**Proposition 2.1.** *Let  $n \in \mathbb{N}$ ,  $\beta \in (0, 1)$ ,*

$$0 < \alpha < \min \left\{ \frac{1}{1 + n(n+1)}, \frac{\beta}{2} \right\},$$

*let  $g_0 \in \mathbb{R}_{\operatorname{sym}}^{n \times n}$  be positive definite and let  $U \subset \mathbb{R}^n$  a smoothly bounded domain. There exist  $\varepsilon_0, C, r > 0$  such that for all  $\theta, \mu, \delta \in (0, \infty)$  satisfying  $\mu \geq \delta$  and  $\delta^{\beta-2} \mu^{-\beta} \theta^2 \leq \varepsilon_0$  the following holds:*

*If  $g \in C^{0,\beta}(\overline{U}, \mathbb{R}_{\operatorname{sym}}^{n \times n})$  satisfies*

$$\|g - g_0\|_0 \leq r \tag{6}$$

$$[g]_\beta \leq \theta^2 \tag{7}$$

*and if  $u \in C^2(\overline{U}, \mathbb{R}^{n+1})$  satisfies*

$$\|u^*e - g\|_0 \leq \delta^2 \tag{8}$$

$$\|\nabla^2 u\|_0 \leq \mu, \tag{9}$$

then there exists an isometric immersion  $\bar{u} \in C^{1,\alpha}(\bar{U}, \mathbb{R}^{n+1})$  of  $g$  with

$$\|\nabla \bar{u} - \nabla u\|_0 \leq C\delta \text{ and } [\nabla \bar{u} - \nabla u]_\alpha \leq C\mu^\alpha \delta^{1-\alpha}.$$

### 3 $h$ -principle for von Kármán constraints

**Proposition 3.1.** *Theorem 1.1 is true provided that, in addition,  $w = 0$ .*

Theorem 1.1 follows at once from Proposition 3.1. For the readers' convenience we include the details:

*Proof of Theorem 1.1.* Applying Proposition 3.1 with  $\tilde{A} = A - 2 \operatorname{sym} \nabla w$ , we obtain  $\bar{v} \in C^{1,\alpha}(\bar{\Omega})$  and  $\tilde{w} \in C^{1,\alpha}(\bar{\Omega}, \mathbb{R}^2)$  satisfying

$$\|\nabla \bar{v} - \nabla v\|_0 \leq C\|\nabla v \otimes \nabla v - \tilde{A}\|_0$$

and

$$\begin{aligned} \|\nabla \tilde{w}\|_{L^p} &\leq C\|\nabla v\|_{C^0(\Omega)}\|\nabla v \otimes \nabla v - \tilde{A}\|_{C^0(\Omega)}^{1/2} \\ &\quad + \|\nabla v \otimes \nabla v - \tilde{A}\|_{C^0(\Omega)} \end{aligned}$$

and  $2 \operatorname{sym} \nabla \tilde{w} + \nabla \bar{v} \otimes \nabla \bar{v} = A - 2 \operatorname{sym} \nabla w$ . Hence by the definition of  $\tilde{A}$  the claim follows with  $\bar{w} = w + \tilde{w}$ .  $\square$

*Proof of Proposition 3.1.* Set  $g_0 = I$ . For every  $t > 0$  define  $g_t = I + t^2 A$ . So  $[g_t]_\beta = t^2[A]_\beta$ . Setting  $\theta_t = \sqrt{1 + [A]_\beta} \cdot t$ , we see that estimate (7) (with index  $t$ ; we omit this remark in what follows) is satisfied. And (6) is satisfied for any  $r > 0$ , provided that  $t < t_0$ , where  $t_0 \in (0, \infty]$  is defined by

$$t_0^2 = \frac{r}{\|A\|_0}.$$

Define  $u_t : \Omega \rightarrow \mathbb{R}^3$  by

$$u_t(x) = \begin{pmatrix} x \\ tv(x) \end{pmatrix}$$

and define  $D \geq 0$  by  $\|\nabla v \otimes \nabla v - A\|_0 = D^2/2$ . We may assume that  $D > 0$ , because if  $D = 0$  then there is nothing to prove. Define  $\delta_t = Dt$ .

We have

$$u_t^* e - g_t = t^2(\nabla v \otimes \nabla v - A).$$

Hence (8) is satisfied.

Finally, for  $M \geq 2(1 + \|\nabla^2 v\|_0 + D)$  and setting  $\mu_t = Mt$ , estimate (9) and  $\mu_t \geq \delta_t$  are satisfied. On the other hand,

$$\delta_t^{\beta-2} \mu_t^{-\beta} \theta_t^2 \leq (1 + [A]_\beta) D^{\beta-2} M^{-\beta} \text{ for all } t > 0. \quad (10)$$

If  $M^\beta$  exceeds  $\varepsilon_0^{-1}(1 + [A]_\beta)D^{\beta-2}$ , then the right-hand side of (10) does not exceed  $\varepsilon_0$ ; here  $\varepsilon_0$  is the constant from Proposition 2.1.

Hence for every  $t \in (0, t_0)$  Proposition 2.1 furnishes isometric immersions  $\bar{u}_t \in C^{1,\alpha}(\bar{\Omega}, \mathbb{R}^3)$  of  $g_t$  satisfying

$$\|\nabla \bar{u}_t - \nabla u_t\|_0 \leq CDt \quad \text{and} \quad [\nabla \bar{u}_t - \nabla u_t]_\alpha \leq CM^\alpha D^{1-\alpha}t. \quad (11)$$

Define  $\bar{\Phi}_t : \Omega \rightarrow \mathbb{R}^2$  and  $\bar{v}_t : \Omega \rightarrow \mathbb{R}$  by

$$\bar{u}_t = \begin{pmatrix} \bar{\Phi}_t \\ t\bar{v}_t \end{pmatrix}.$$

Then (11) imply that

$$\begin{aligned} [\nabla \bar{v}_t - \nabla v]_\alpha &\leq CM^\alpha D^{1-\alpha} \\ \|\nabla \bar{v}_t - \nabla v\|_0 &\leq CD \end{aligned} \quad (12)$$

for all  $t \in (0, t_0)$ . And  $\|\nabla \bar{\Phi}_t - I\|_0 \leq CDt$ . In particular,  $\det \nabla \bar{\Phi}_t > 0$  for  $t > 0$  small enough. Moreover, since  $\bar{u}_t^* e = I + t^2 A$ ,

$$(\nabla \bar{\Phi}_t)^T (\nabla \bar{\Phi}_t) = I + t^2 (A - \nabla \bar{v}_t \otimes \nabla \bar{v}_t). \quad (13)$$

Hence

$$\|\sqrt{\nabla \bar{\Phi}_t^T \nabla \bar{\Phi}_t} - I\|_{L^p} \leq \|\nabla \bar{\Phi}_t^T \nabla \bar{\Phi}_t - I\|_{L^p} \leq t^2 \|A - \nabla \bar{v}_t \otimes \nabla \bar{v}_t\|_{L^p}.$$

Since  $\det \nabla \bar{\Phi}_t > 0$ , we have almost everywhere

$$\text{dist}_{SO(2)}(\nabla \bar{\Phi}_t) = \left| \sqrt{(\nabla \bar{\Phi}_t)^T (\nabla \bar{\Phi}_t)} - I \right|.$$

Hence by FJM-rigidity (cf. [2, Theorem 3.1] and the sentence following its statement) there exists a constant  $C$  depending only on  $p$  (and on  $\Omega$ ) and there exist  $R_t \in SO(2)$  as well as  $\bar{w}_t \in W^{1,p}(\Omega, \mathbb{R}^2)$  such that, for  $t > 0$  small enough,

$$\|\nabla \bar{w}_t\|_{L^p} \leq C \|A - \nabla \bar{v}_t \otimes \nabla \bar{v}_t\|_{L^p},$$

and such that

$$\nabla \bar{\Phi}_t = R_t + t^2 \nabla \bar{w}_t.$$

Denoting by  $\tilde{R}_t \in SO(3)$  the matrix with rotation axis  $(0, 0, 1)^T$  and in-plane rotation  $R_t$ , define

$$\tilde{u}_t = \tilde{R}_t^T \bar{u}_t$$

and  $\tilde{w}_t = R_t^T \bar{w}_t$ . Then

$$\nabla \tilde{u}_t = \begin{pmatrix} I + t^2 \nabla \tilde{w}_t \\ t \nabla \bar{v}_t \end{pmatrix}$$

and

$$\|\nabla \tilde{w}_t\|_{L^p} \leq C \|A - \nabla \bar{v}_t \otimes \nabla \bar{v}_t\|_{L^p}. \quad (14)$$

Since  $\tilde{u}_t$  is an isometric immersion of  $g_t$ , we have

$$2 \operatorname{sym} \nabla \tilde{w}_t + t^2 (\nabla \tilde{w}_t)^T (\nabla \tilde{w}_t) = A - \nabla \bar{v}_t \otimes \nabla \bar{v}_t. \quad (15)$$

By (12) and (14) there exists a sequence  $t \rightarrow 0$  and  $\bar{v} \in C^{1,\alpha}$  such that  $\nabla \bar{v}_t \rightarrow \nabla \bar{v}$  uniformly and such that  $\nabla \tilde{w}_t$  converges weakly in  $L^p$  to the gradient of some  $\bar{w} \in W^{1,p}$ . By (14), the matrix fields  $(\nabla \tilde{w}_t)^T (\nabla \tilde{w}_t)$  remain uniformly bounded in  $L^1$  as  $t \rightarrow 0$ . Hence letting  $t \rightarrow 0$  in (15), we conclude that

$$2 \operatorname{sym} \nabla \bar{w} = A - \nabla \bar{v} \otimes \nabla \bar{v}.$$

Moreover, taking the limes inferior in (14), we have

$$\|\nabla \bar{w}\|_{L^p} \leq C \|A - \nabla \bar{v} \otimes \nabla \bar{v}\|_{L^p}.$$

And by (12)

$$\|A - \nabla \bar{v} \otimes \nabla \bar{v}\|_0 \leq C \|A - \nabla v \otimes \nabla v\|_0 + C \|\nabla v\|_0 \|A - \nabla v \otimes \nabla v\|_0^{1/2}.$$

□

Combining Theorem 1.1 with a Nash-Kuiper result one obtains the following; see [4] for a similar result.

**Corollary 3.2.** *Let  $\beta$ ,  $\alpha$  and  $A$  be as in Theorem 1.1. Let  $v \in C^1(\bar{\Omega})$  and  $w \in C^1(\bar{\Omega}, \mathbb{R}^2)$  be such that*

$$2 \operatorname{sym} \nabla w + \nabla v \otimes \nabla v \leq A - cI \text{ as symmetric matrices,}$$

*for some constant  $c > 0$ , and let  $\varepsilon > 0$ . Then there exist  $\bar{v} \in C^{1,\alpha}(\bar{\Omega})$  and  $\bar{w} \in C^{1,\alpha}(\bar{\Omega}, \mathbb{R}^2)$  with*

$$\|\bar{w} - w\|_{C^0(\Omega, \mathbb{R}^2)} + \|\bar{v} - v\|_{C^0(\Omega)} \leq \varepsilon$$

*such that*

$$2 \operatorname{sym} \nabla \bar{w} + \nabla \bar{v} \otimes \nabla \bar{v} = A \text{ on } \Omega.$$

*Proof.* Let  $\delta \in (0, 1)$  and let  $p \in (2, \infty)$ . Following [1], let  $\tilde{v}$  respectively  $\tilde{w} \in C^\infty(\overline{\Omega})$  be  $C^1$ -close to  $v$  respectively  $w$ . Applying [4, Theorem 2.1] with  $\tilde{v}$ ,  $\tilde{w}$  and some smooth uniform approximation of  $A$ , one obtains  $\hat{v}$ ,  $\hat{w} \in C^1$  such that

$$\|\hat{w} - w\|_0 + \|\hat{v} - v\|_0 + \|2 \operatorname{sym} \nabla \hat{w} + \nabla \hat{v} \otimes \nabla \hat{v} - A\|_0 \leq \delta^2.$$

By approximation, we may assume that  $\hat{v}$ ,  $\hat{w} \in C^2(\overline{\Omega})$ . Applying Theorem 1.1 we obtain  $\overline{v}$ ,  $\overline{w} \in C^{1,\alpha}(\overline{\Omega})$  satisfying

$$\|\nabla \overline{v} - \nabla \hat{v}\|_0 \leq C \|\nabla \hat{v} \otimes \nabla \hat{v} + 2 \operatorname{sym} \nabla \hat{w} - A\|_0^{1/2} \leq C\delta$$

and

$$\|\nabla \overline{w} - \nabla \hat{w}\|_{L^p} \leq C(\delta + \|\nabla \hat{v}\|_0)\delta,$$

and  $2 \operatorname{sym} \nabla \overline{w} + \nabla \overline{v} \otimes \nabla \overline{v} = A$ . Notice that  $\hat{v}$  can be chosen such that  $\|\nabla \hat{v}\|_0 \leq C(1 + \|\nabla \tilde{v}\|_0) \leq C(1 + \|\nabla v\|_0)$  for some constant  $C$  independent of  $\delta$ , cf. [4, Remark 3.3].

The claim now follows from the continuous embedding of  $W^{1,p}$  into  $C^0$ , and from the arbitrariness of  $\delta$ .  $\square$

**Acknowledgements.** Both authors acknowledge support by the DFG.

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